

Hyperelastic Material Models for Finite Element Analysis of Rubber[†]

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Traditional design of rubber engineering components involves hand calculations assuming linear (Hookean) stress-strain behaviour. Such calculations are restricted to the simplest geometries and ignore the well-known non-linear stress-strain behaviour of rubber. Modern design methodology based on finite element analysis do not suffer from these limitations. However, the use of such computer-based techniques is hampered by difficulties in describing the stress-strain behaviour of rubber in the form of suitable hyperelastic material models. This is in spite of the fact that the underlying mathematics have been well developed by the pioneering efforts of Mooney, Rivlin, and Ogden.

This paper describes recent work that has yielded new perspectives of the Rivlin and Ogden hyperelastic material models. It suggests that rubber is best represented by a model where the shear modulus varies with strain in a relatively simple manner. While the approach has been developed as a practical solution to the problem of material characterisation for the purposes of finite element analysis, it has interesting repercussions in our interpretation of the molecular basis for rubber elasticity.

In the traditional method of designing rubber engineering components^{1,2}, the designer uses simple 'hand' calculations to estimate load-deflection characteristics and load-bearing capacity. To keep the calculations tractable, it is necessary to assume that the rubber material exhibits *linear* (Hookean) stress-strain behaviour, ignoring the well-known *non-linear* characteristics of rubber elasticity. Even so, such calculations are limited to the simplest of geometries and loading conditions. As a result, the calculations are at best only approximate and there are considerable reliance on experience and somewhat arbitrary 'rules of thumb'. This method produces rather

conservative designs which have served us quite well up to now.

Today, however, the customer is constantly demanding more efficient designs. Modern components must be smaller, lighter, carry more load, serve under harsher environments, last longer, *etc.* and yet be cheaper than the part it replaces. Also, the ever shortening of the new product cycle means that there is less and less time available for multiple prototypes and extended testing. The traditional method of hand calculations is simply not accurate enough for us to get it right the first time. We need more accurate predictions of performance. We

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cannot ignore the non-linearity of rubber just to keep the mathematics simple.

Fortunately, intractable calculations can often be solved numerically. When engineers first applied finite element analysis (a computer-based technique for engineering analysis) to rubber design problems³ in the early 1970s, only the largest universities and research institutes could afford the powerful computers needed to tackle even the simpler problems. Today, such computer facilities are readily available to everyone. So, it is no surprise to find that finite element analysis is fast becoming the normal method for designing rubber components.

But, computer hardware and finite element analysis software are not enough. They are merely tools for the skilled engineer which enable him to analyse the problem more accurately and quickly. There is no substitute for the skilled engineer's knowledge and understanding of his materials, processes, designs and service requirements. We shall only discuss one facet of the materials knowledge base here; the stress-strain properties of rubber.

Finite element analysis of rubber components requires as input a mathematical description of the stress-strain properties of the material. This usually takes the form of a strain energy function sometimes referred to as a 'hyperelastic material model'. Characterisation of a specific rubber material consists of performing stress-strain measurements followed by curve-fitting the data. Finite element analysis programs even provide utilities for the regression analysis. However, the task is not as trivial as it seems. Inadequate material models are often the cause of lack of agreement between finite element analysis predictions and experiment. This paper is intended to serve as

a practical guide to the novice on how reasonable, even if approximate, material models may be obtained from a limited testing program.

THEORETICAL FOUNDATIONS

Material models may be derived from two separate approaches to the study of rubber elasticity: (a) the statistical or kinetic theory; and (b) the phenomenological theory.

Statistical Theory

According to the Statistical Theory of Rubber Elasticity⁴, the non-linear stress-strain behaviour of rubber may be derived from molecular considerations to yield the constitutive equation:

$$W = C_{10}(I_1 - 3) \quad \dots 1$$

where W is the strain energy density, C_{10} is a material constant which is related to molecular parameters, and I_1 is the first invariant of the Green deformation tensor. Equation 1 is commonly known as the neo-Hookean material model. Unfortunately, this model is only valid for a relatively small range of strains⁴. Many attempts have been made to refine the basic Statistical Theory. These yield increasingly complex constitutive models but none have found practical utility in finite element analysis. Nevertheless, the Statistical Theory occupies a central position in our understanding of the molecular basis of rubber elasticity.

Phenomenological Theory

Mooney⁵ took the phenomenological approach which treats the problem from the continuum mechanics viewpoint. Assuming that

the rubber is homogenous, isotropic, elastic, incompressible and obeys Hooke's law in simple shear, he obtained the strain energy function:

$$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) \quad \dots 2$$

where C_{10} and C_{01} are material constants and I_1 and I_2 are the first two invariants of the Green deformation tensor. Equation 2 is usually quite successful in describing tensile data up to moderately large strains of about 100%. However, it is significantly less successful in other modes of deformation, especially compression. In spite of this limitation, it is probably the most commonly used material model in the finite element analysis of rubber. Although Mooney's phenomenological approach makes no appeal to molecular theory, it is common to identify C_{10} with molecular parameters in the Statistical Theory and to interpret C_{01} as a measure of deviations from the theory perhaps reflecting a failure to achieve equilibrium conditions⁶.

Rivlin⁷ considerably expanded and generalised Mooney's approach and showed that the most general strain energy function for a homogenous, isotropic, incompressible, elastic material is:

$$W = \sum_{i+j=1}^n C_{ij} (I_1 - 3)^i (I_2 - 3)^j \quad \dots 3$$

The first order approximation of this power series is identical to Equation 2 which became popularly known as the Mooney-Rivlin model.

More recently, another material model due to Ogden⁸ appears to be gaining popularity. Ogden's strain energy function may be written as:

$$W = \sum_{p=1}^n \frac{\mu_p}{\alpha_p} (\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3) \quad \dots 4$$

where μ_p and α_p are material constants and λ_1 , λ_2 , and λ_3 , are the principal extension ratios. In the Ogden formulation, the indices, α_p , need not be integers.

We note that the Rivlin and Ogden material models are, power series and so it appears possible to fit experimental stress-strain data to any desired degree of precision by merely taking a sufficient number of terms. Therefore, these material models are in principle better than the two-constant Mooney-Rivlin model. Indeed, it is a common fallacy to assume that the more terms in the strain energy function, the better. In practice, the additional degrees of freedom introduced by extra terms allow the regression analysis to do a better job of fitting experimental errors in the stress-strain data! The result is often unstable strain energy functions which predict physically unrealistic behaviour under conditions outside the range of experimental data⁹⁻¹¹. Indeed, higher order strain energy functions usually show such poor ability to predict behaviour outside the range of experimental data that Chow and Cundiff¹¹ recommended the use of the Mooney-Rivlin material model. Clearly, its simplicity and robustness are so valued that its inaccuracies are tolerated.

According to Rivlin and Sawyers¹², the Ogden model is a special case of the Rivlin strain energy function. Treloar⁴ expressed the opinion that the two formulations were essentially equivalent and that the choice of one over the other is simply a question of convenience. So, from the viewpoint of performing finite element analysis, it does not matter which strain energy function is used so long as it gives an adequate representation of

the material properties with a reasonable number of fitting coefficients. The question that remains is how the coefficients of the chosen material model may be obtained from a modest testing program.

RECENT DEVELOPMENTS

One difficulty common to both the Rivlin and Ogden material models is the fact that they contain a number of arbitrary fitting constants of uncertain physical significance. So long as we regard the development of material models merely as a curve-fitting exercise, it is difficult to determine when an adequate model has been derived and further refinements are not justified. We review here some recent work which has shed new light on the underlying physics of these models. Armed with this new understanding, the development of realistic material models is seen to be significantly simplified.

Rivlin's Formulation

We start by considering a material obeying Rivlin's model (Equation 3) subjected to simple shear deformation. The shear stress, τ , is related to the amount of shear, γ by⁴

$$\frac{\tau}{\gamma} = 2 \left[\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2} \right] \quad .5$$

It is seen that the shear modulus is given by the sum of the partial derivatives $\partial W/\partial I_1$ and $\partial W/\partial I_2$. For the specific cases of the neo-Hookean and Mooney-Rivlin material models, $\partial W/\partial I_1$ and $\partial W/\partial I_2$ are constants; that is the shear modulus is constant. However, in the more general Rivlin material model, $\partial W/\partial I_1$ and $\partial W/\partial I_2$ are functions of I_1 and I_2 . We note from

Equation 5 that it is impossible to evaluate the individual values of $\partial W/\partial I_1$ and $\partial W/\partial I_2$ from shear stress-strain data. In fact, Kawabata and Kawai¹³ have pointed out that it is impossible to evaluate $\partial W/\partial I_1$ and $\partial W/\partial I_2$ from stress-strain data from any single mode of deformation, it is necessary to have data from two or more modes of deformation. This increases considerably the amount of experimental work needed to characterise the rubber.

A considerable simplification is possible. Gregory¹⁴ had noted that usually $\partial W/\partial I_1$ is much larger than $\partial W/\partial I_2$. So, Yeoh¹⁵ suggested assuming $\partial W/\partial I_2 = 0$. This simplifies Rivlin's material model to:

$$W = \sum_{i=1}^{\infty} C_i (I_1 - 3)^i \quad 6$$

where j is always zero but has been retained for consistency with Rivlin's nomenclature. The shear modulus, τ/γ , is then given by.

$$\frac{\tau}{\gamma} = 2 \frac{\partial W}{\partial I_1} = 2C_{10} + 4C_{20}(I_1 - 3) + 6C_{30}(I_1 - 3)^2 + \dots \quad 7$$

Recalling that the shear strain is related to I_1 by⁴

$$\gamma^2 = (I_1 - 3) \quad .8$$

it is seen that Yeoh's model implies a material whose shear modulus varies with even powers of the amount of shear. This is consistent with the symmetry of the deformation.

Figure 1 shows the dependence of shear modulus on $(I_1 - 3)$ for a typical unfilled natural rubber vulcanisate. We find that at small strains, the shear modulus decreases with increasing

strain but at large strains, the shear modulus increases with increasing strain. A convenient way to describe this strain dependence of shear modulus is the relation^{16,17}:

$$\frac{\tau}{\gamma} = 2A \exp[-B(I_1 - 3)] + 2C_{10} + 4C_{20}(I_1 - 3) + 6C_{30}(I_1 - 3)^2 + \dots \dots 9$$

which implies the strain energy function:

$$W = \frac{A}{B} \left\{ 1 - \exp[-B(I_1 - 3)] \right\} + \sum_{i=1}^{\infty} C_{i0} (I_1 - 3)^i \quad \dots 10$$

Since the exponential term may be written as a polynomial, it is seen that *Equations 9 and 10* are not really different from *Equations 7 and 6*. But this particular form has the advantage of implying two mechanisms; the exponential term reflects the small strain behaviour where the shear modulus decreases with increasing strain while the polynomial term reflects the large strain behaviour where the shear modulus increases with increasing strain.

Expressing the strain energy function in terms of $(I_1 - 3)$ alone, simplifies the problem of material characterisation considerably since it is no longer necessary to evaluate $\partial W/\partial I_1$ and $\partial W/\partial I_2$ at fixed values of I_2 and I_1 , respectively. Also, since $\partial W/\partial I_2$ is now irrelevant, data from any single deformation mode is sufficient. Regression analysis using shear data is obvious from the above discussion but the more readily available tensile data may be used instead¹⁵. Thus, it is possible to develop appropriate strain energy functions to describe elastic behaviour of rubbers from a limited test program^{15,16}. Strain energy functions of this

form (*Equations 6 and 10*) have been used successfully in finite element analysis¹⁸⁻²⁰

Ogden's Formulation

Here, we seek some physical interpretation of the material parameters, μ_p and α_p . For simplicity and compact printing, we shall consider an Ogden strain energy function with just one term and accordingly will drop the subscript in the following. The arguments can be easily extended to Ogden models with n terms since each term has the same form.

We start by considering simple shear deformation. It can be readily shown²¹ that the secant shear modulus is given by:

$$\frac{\tau}{\gamma} = \frac{2G_0}{\alpha} \left[\frac{\lambda^\alpha - \lambda^{-\alpha}}{\lambda^2 - \lambda^{-2}} \right] \quad \dots 11$$

where G_0 is the small strain shear modulus, and λ is the extension ratio of the equivalent pure shear deformation which is related to the amount of shear, γ by:

$$\gamma = \lambda - \lambda^{-1} \quad \dots 12$$

Equation 11 shows that the secant shear modulus is a function of α and of the shear strain (which is reflected in λ). The form of the dependence is shown in *Figure 2* where the shear stress, τ , is plotted as a function of shear strain, γ , for materials with a small strain shear modulus, G_0 , equal to 1 MPa but with different values for the index α . It is seen that when α takes the value of 2 (*i.e.* the material is neo-Hookean), the shear stress-strain curve is linear (*i.e.* the shear modulus is constant). When α is less than 2, the material softens with increasing strain. On the other hand, when

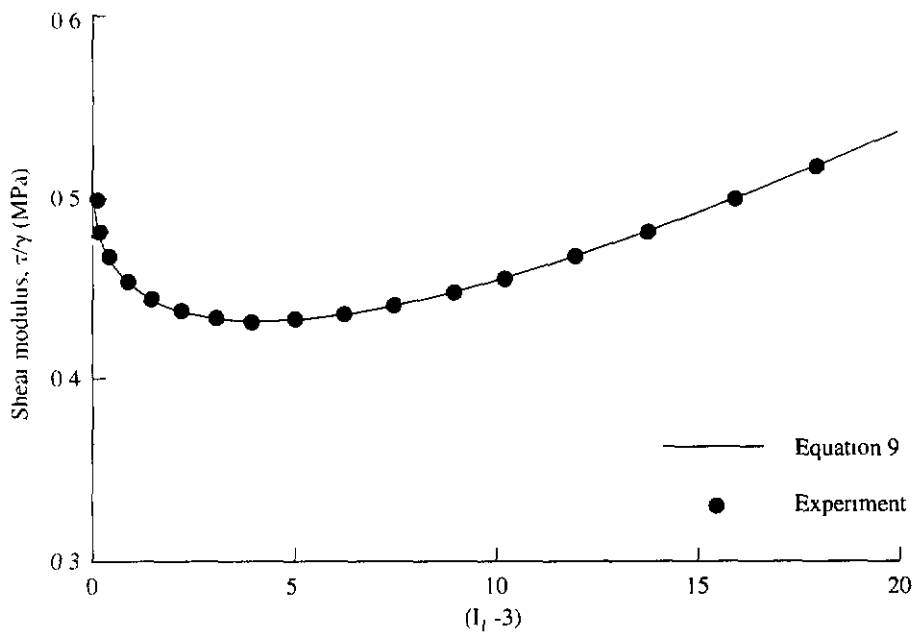


Figure 1 Strain dependence of shear modulus for typical unfilled natural rubber vulcanisate

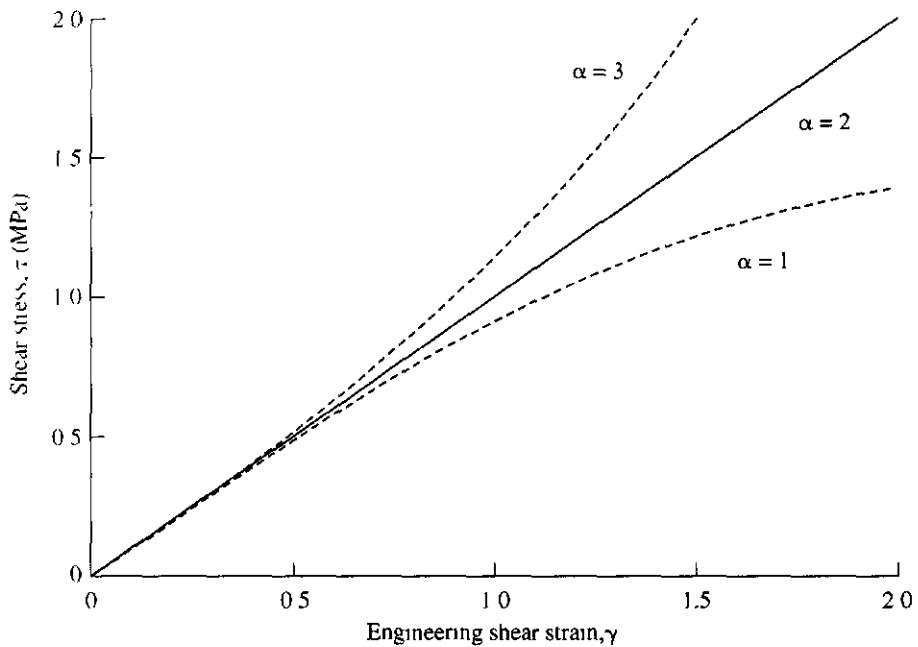


Figure 2 Simple shear stress-strain curves for Ogden materials with different values of α

α is *greater* than 2, the material *stiffens* with increasing strain. The form of Equation 11 is such that the same behaviour is obtained when α takes a negative value; it is the absolute value of α which determines whether the shear modulus decreases or increases with increasing strain. Although this dependence of shear modulus on α is intrinsic to the Ogden strain energy function, it is strange that there appears to be no comment about this in the literature.

When we consider Ogden material models with $n > 1$, we note that the product $\mu_p \alpha_p$ each makes a contribution to the shear modulus and that G_0 is given by:

$$G_0 = \sum_{p=1}^n \frac{\mu_p \alpha_p}{2} \quad \dots 13$$

For the Mooney-Rivlin material model, $\alpha_1 = 2$ and $\alpha_2 = -2$. From the above we find that the shear modulus of the Mooney-Rivlin model is constant. This, of course, is no surprise since Mooney⁵ started with the assumption that the stress-strain curve of rubber is linear in simple shear. When $|\alpha_p| < 2$, that term's contribution to the secant shear modulus decreases with increasing strain and conversely, when $|\alpha_p| > 2$, that term's contribution to the shear modulus increases with increasing strain. So, the shear stress-strain curve of a multi-term Ogden material will meander to a greater or lesser degree depending on the relative magnitudes of α_p and μ_p .

Ogden suggested that stress-strain data for an unfilled natural rubber vulcanisate reported by Treloar²² may be represented by an Ogden material model with the coefficients:

$$\begin{aligned}\alpha_1 &= 1.3, \mu_1 = 0.618 \text{ MPa} \\ \alpha_2 &= 5.0, \mu_2 = 0.00118 \text{ MPa} \\ \alpha_3 &= -2.0, \mu_3 = -0.00981 \text{ MPa}\end{aligned}$$

From the above, it is now clear what Ogden's model implies. The first term ($\alpha_1 = 1.3$) indicates that the shear modulus *decreases* with increasing strain. The relatively large value of μ_1 means that this term dominates behaviour at small strain. The second term ($\alpha_2 = 5.0$) indicates that the shear modulus *increases* with increasing strain and this term will dominate behaviour at large strains. The third term ($\alpha_3 = -2.0$) reflects on relative contributions to tensile and compressive stresses when the deformation deviates from shear²¹. The relatively small value of μ_3 means that this term is not very significant and neglecting it is equivalent to the neglect of the $(I_2 - 3)$ by Yeoh. Thus, taken together, Ogden's model describes the same general dependence of shear modulus on strain we discussed earlier.

IMPLICATIONS FOR MOLECULAR THEORY

Although our primary motivation for the study of the phenomenological theory of rubber elasticity is the development of material models to facilitate finite element analysis, we cannot help but reflect upon the implications for the molecular theory. A material model with arbitrary coefficients remains nothing more than a curve-fit unless we are able to relate the coefficients to some fundamental principle. Some recent attempt has been made in this direction.

We note that the elementary Statistical Theory leading to Equation 1 assumes Gaussian chain statistics which are invalid as the chains approach their fully extended state at large

strains. More elaborate theories have been developed taking into account the finite extensibility of the chains. Specifically, Langevin chain statistics have been invoked⁴. Wang and Guth²³ showed that consideration of finite chain extensibility leads to additional, higher order terms in I_1 and I_2 . Unfortunately, the resulting strain energy function depends on the specifics of the molecular model and the number of terms in the material model makes it difficult to make direct comparisons between theory and experiment.

Recently, Gent²⁴ suggested a new strain energy function which may be written as:

$$W = -C_{10} (I_m - 3) \ln \left[1 - \frac{(I_1 - 3)}{(I_m - 3)} \right] \quad \dots 14$$

where I_m is the limiting value of I_1 corresponding to the deformation when the network is fully stretched. The logarithmic term may be written as a polynomial, so Equation 14 is of the form of Equation 6. Ignoring I_2 terms may be partly justified on the grounds that at least in the cases of certain molecular network models proposed by Wang and Guth²³ and Arruda and Boyce²⁵, strain energy functions have been derived which contain only terms in I_1 . Equation 14 predicts a shear modulus which increases with increasing strain. This is consistent with observed behaviour at large strains. So, it appears that the increase in modulus with increasing strain is associated with finite chain extensibility.

Gent's strain energy function has the advantage of reducing rubbery stress-strain behaviour to just two closely related parameters which, moreover, have clear physical meaning

in molecular terms. The constant I_m is the limiting value of I_1 corresponding to the deformation when the network is fully-stretched. So I_m is related to chain length and inversely related to crosslink density. Recalling from the Statistical Theory that C_{10} is proportional to crosslink density, we see that I_m is inversely proportional to C_{10} .

Recently, Yeoh and Fleming¹⁷ extended Gent's treatment and suggested that the same parameters have significance at small strains too. They suggested writing the strain energy function in the form:

$$W = \frac{A}{B} (I_m - 3) [1 - \exp(-BR)] - C_{10} (I_m - 3) \ln(1 - R) \quad \dots 15$$

where:

$$R = \frac{(I_1 - 3)}{(I_m - 3)} \quad \dots 16$$

and presented supporting experimental data. They tested a family of four unfilled natural rubber vulcanisates which differ in crosslink density. Figure 3 shows that $(I_m - 3)$ is inversely proportional to C_{10} as expected from Gent's theory. Also, the stress-strain behaviour of all four rubbers in different deformation modes may be predicted from measurements made on just one member in a single mode. This is illustrated in Figure 4 where the tensile stress-strain curve of one member, shown as a solid line, has been fitted to Equation 15 and the derived material coefficients have been used to predict the behaviour of the entire family under different deformation modes. The predictions are shown in Figures 4-6 as broken

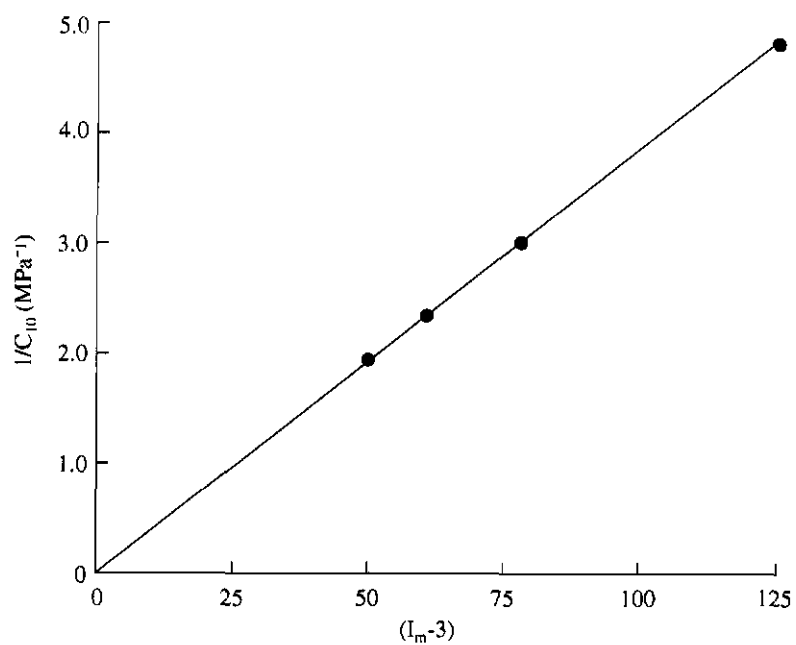


Figure 3. Relation between $(I_m - 3)$ and $1/C_{10}$.

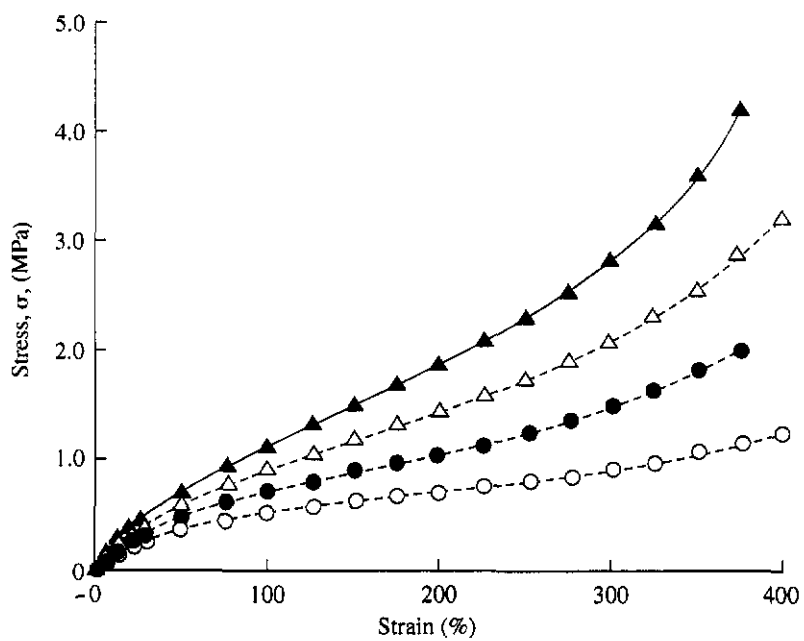


Figure 4. Comparison of theory with experiment. Simple extension.

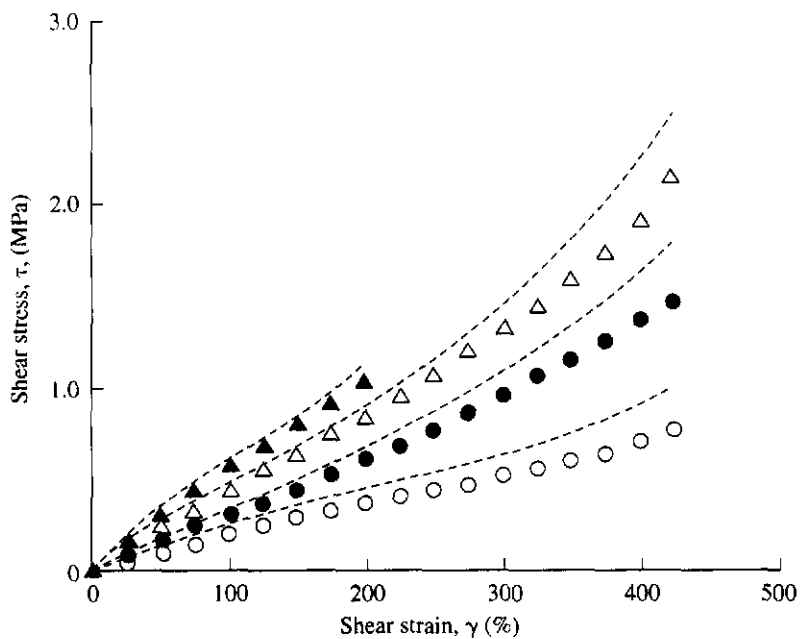


Figure 5. Comparison of theory with experiment. Simple shear.

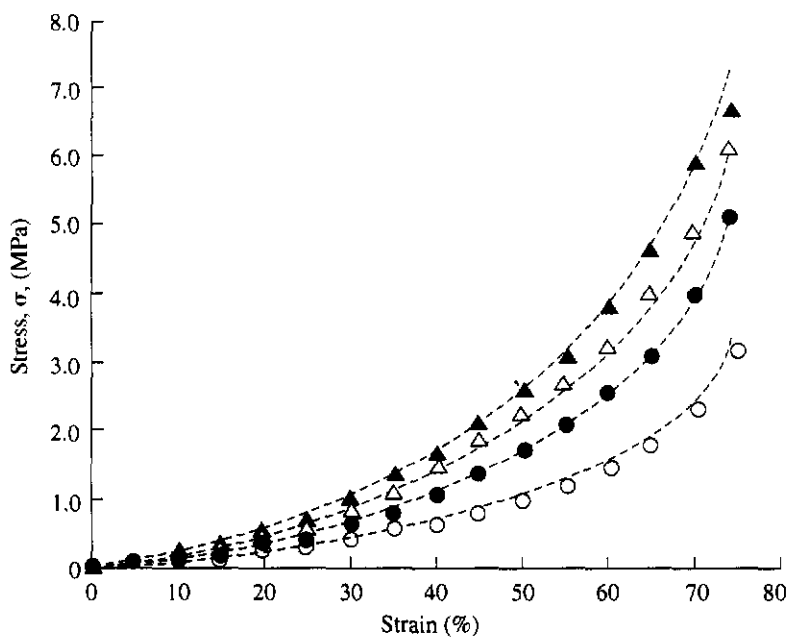


Figure 6. Comparison of theory with experiment. Compression between lubricated plates.

lines. It is seen that the agreement between theory and experiment is quite satisfactory.

CONCLUSIONS

We conclude that the stress-strain properties of rubber is best represented by a model where the shear modulus varies with strain in a relatively simple manner. At small strains, the shear modulus *decreases* with increasing strain. At large strains, the shear modulus *increases* with increasing strain. This insight has facilitated the development of practical material models for finite element analysis.

This approach has interesting repercussions in our interpretation of the molecular basis for rubber elasticity. We regard the model from classical Statistical Theory as our basic model with two mechanisms for enhancement of shear modulus. At large strains, the shear modulus increases because of finite chain extensibility. At small strains, the source of shear modulus enhancement is less obvious. Empirically, we know that it decreases rapidly from a finite value as strain increases. We speculate that it may be related to network defects. Unlike previous treatments which interpret deviations from the Statistical Theory in terms of the second Mooney constant (*i.e.* in terms of I_2), we attribute all deviations to additional terms in I_1 .

REFERENCES

1. LINDLEY, P.B. (1974) Engineering Design with Natural Rubber. *Technical Bulletin*, Malaysian Rubber Producers' Research Association, Brickendonbury, England.
2. FREAKLEY, P.K. AND PAYNE, A.R. (1978) *Theory and Practice of Engineering with Rubber*. London: Applied Science.
3. LINDLEY, P.B. (1973) *Proc. Intl. Rubb. Conf., Brighton, May 1972*. London: Institution of the Rubber Industry.
4. TRELOAR, L.R.G. (1975) *The Physics of Rubber Elasticity*, 3rd ed. Oxford: Clarendon Press.
5. MOONEY, M. (1940) *J. Appl. Phys.*, **11**, 582.
6. KRIGBAUM, W.R. AND ROE, R-J (1965) *Rubb. Chem. Technol.*, **38**, 1039.
7. RIVLIN, R.S. (1956) Large Elastic Deformations. *Rheology: Theory and Applications*, (Eirich, F.R. ed.) Vol. 1. N.Y.: Academic Press.
8. OGDEN, R.W. (1972) *Proc. R. Soc. Lond., A* **326**, 565.
9. JAMES, A.G., GREEN, A. AND SIMPSON, G.M. (1975) *J. Appl. Polym. Sci.*, **19**, 2033.
10. JAMES, A.G. AND GREEN, A. (1975) *J. Appl. Polym. Sci.*, **19**, 2319.
11. CHOW, C.L. AND CUNDIFF, C.H. (1987) *Tire Sc. Technol.*, **15**, 73.
12. RIVLIN, R.S. AND SAWYERS, K.N. (1976) *Trans. Soc. Rheo.*, **20**, 545.
13. KAWABATA, A. AND KAWAI, H. (1977) *Adv. Polym. Sci.*, **24**, 89.
14. GREGORY, M.J. (1979) *Plast. Rubber Mater. Appl.*, **4**, 184.
15. YEOH, O.H. (1990) *Rubb. Chem. Technol.*, **63**, 792.
16. YEOH, O.H. (1993) *Rubb. Chem. Technol.*, **66**, 754.
17. YEOH, O.H. AND FLEMING, P.D. (1997) *J. Polym. Sci., Part B*, **35**, 1919.
18. CHEN, J.S., SATYAMURTHY, K. AND HIRSCHFELT, L.R. (1994) *Computer and Structures*, **50**, 715.

- 19 CHEN, J S AND PAN, C (1996) *J Appl Mechanics*, **68**, 862
- 20 CHEN, J S, WU, C T AND PAN, C (1996) *J Appl Mechanics*, **68**, 869
- 21 YEOH, O H (1997) *Rubb Chem Technol*, **70**, 175
- 22 TRELOAR, L R G (1994) *Trans Faraday Soc*, **40**, 59
- 23 WANG, M C AND GUTH, E (1952) *J Chem Phys*, **20**, 1144
- 24 GENT, A N (1996) *Rubb Chem Technol*, **69**, 59
- 25 ARRUDA, E M AND BOYCE, M C (1993) *J Mech Phys Solids*, **41**, 389