The Compression Modulus of Tall Rubber Cylinders

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An approximate theoretical analysis has been developed that enables the evaluation of the compression modulus of tall cylinders bonded to rigid end plates. Comparison is made with other theoretical and numerical solutions and with experimental data. The analysis is also applicable to the case of a cylinder bonded at one end and lubricated at the other. Further, the approach provides an explanation as to why cavitation occurs near the ends, rather than in the middle, when the cylinder is under tension.

The widespread use of bonded rubber blocks as springs and mountings makes it desirable to have an exact knowledge of their elastic behaviour. One particularly important case—bonded cylinders subjected to small axial compressions—has been the subject of several theoretical and experimental studies.

Using an approximate theoretical analysis and assuming that the rubber is incompressible, Gent and Undley1 derived the relation

\[ E_c = E_O (1 + \frac{a^2}{2h^2}) \] \quad \ldots \text{Equation 1} \]

where

- \( E_c \) is the apparent compression modulus of the bonded cylinder
- \( E_O \) is the Young's modulus of the rubber material
- \( a \) is the radius of the cylinder
- \( h \) is the height of the cylinder.

For unfilled natural rubber (NR) vulcanisates, Equation 1 has been shown to fit experimental data over a substantial range of radius/height ratios. The agreement for filled rubbers is poorer and an empirical relation\(^1, 2\) of the form

\[ E_c = E_O (1 + \frac{\beta a^2}{h^2}) \] \quad \ldots \text{Equation 2} \]

where \( \beta \) is a numerical factor, is commonly used.

Lindley\(^3\) has extended this approximate theoretical analysis to compressible materials while Gent and Meinecke\(^4\) have extended it to blocks of any cross-sectional shape.

While Equations 1 and 2 fit experimental data fairly well and are adequate for

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most practical purposes, there have been efforts to obtain more exact solutions. Moghe and Neff developed a theoretical solution and pointed out the inadequacies of Equations 1 and 2 particularly at the extreme ends of the range of radius/height ratios (i.e. for tall and short cylinders). Lindley and Teo have used finite element analysis to obtain the compression modulus numerically. Comparing the numerical solution to the approximate theoretical solution, Lindley found the largest discrepancy occurred with tall cylinders.

Unfortunately, both the Moghe/Neff and Lindley/Teo refinements involve extensive computing and hence are not particularly convenient to use. An approximate analysis is presented here for the case of tall cylinders. This analysis gives a relation which agrees well with Lindley and Teo's numerical results but has the advantage that it is expressed in a simple analytical form.

**THEORETICAL CONSIDERATIONS**

**Tall Cylinders Bonded at Both Ends**

When a tall cylinder bonded to rigid plates at its flat ends is compressed axially, the deformation of the rubber near the bonded surfaces is complex because of the constraining effect of the rigid plates. However, according to St. Venant's principle, the effect is localised and the central portion of the cylinder, away from the ends, will experience only simple, homogeneous compression**.

Thus, conceptually, the bonded cylinder is regarded as three springs in series (Figure 1) and we write

\[
\frac{1}{K} = \frac{1}{K_a} + \frac{2}{K_b}
\]

where \( K \) is the overall stiffness of the cylinder, \( K_a \) is the stiffness of the central portion and \( K_b \) is the stiffness of each end-portion.

![Figure 1. Schematic diagram of tall, bonded cylinder under compression.](image)

**This can only hold if the cylinder is sufficiently tall. For short cylinders, all the rubber will be near a bonded surface and no part will be in simple homogeneous compression.**
For cylinders of different heights, the length of the central portions will vary and hence $K_a$ will vary but so long as such a central region exists, the end portions are not affected and $K_b$ will remain constant. $K_b$ may be determined experimentally by comparing the stiffnesses of bonded cylinders of different lengths.

The magnitude of the end-effect may also be calculated as follows:

Assuming that the rubber is incompressible and that the free surface of the rubber in the end portions bulges out to take up a parabolic shape*** under compression, it follows from geometrical considerations that

$$k = \frac{3}{4} e' a$$

where $k$ is the maximum outward displacement of the free surface,

$e'$ is the apparent compressive strain in the end-portions

$a$ is the radius of the cylinder.

If $e$ is the compressive strain in the central portion, then $k$ is also equal to $\frac{ae}{2}$.

Hence $e' = 2e/3$ and the effective modulus of each end-portion is $3E_o/2$.

When the length of the central portion is zero, the whole of the bulging free surface takes the shape of a parabola and our three-part cylinder reduces to the case considered by Gent and Lindley. $Equation 1$ will then apply. Substituting $3E_o/2$ for $E_c$ in $Equation 1$ yields the result that the length of the two end-portions together is $a$ (i.e. the length of each end-portion is $a/2$).

The overall apparent strain is

$$e'a + e(h-a)$$

where $h$ is the height of the cylinder. This is equal to $(1 - a/3h)e$. Hence the apparent modulus $E_c$ is given by

$$E_c = \frac{E_o}{(1 - \frac{a}{3h})} \quad \ldots Equation 3$$

Cylinders Bonded at One End and Lubricated at the Other

Although the case of a cylinder bonded at one end and compressed against a lubricated plate at the other end is of some interest from a practical viewpoint, it does not appear to have been treated previously. Provided the cylinder is not too short, the same argument as above may be used and this leads to an apparent modulus given by

$$E_c = \frac{E_o}{(1 - \frac{a}{6h})} \quad \ldots Equation 4$$

Comparison with Other Treatments and Experiments

$Equation 3$ is compared with other treatments and experimental data in Figure 2. There is only a small difference between $Equation 3$ (full line) and Gent/Lindley's $Equation 1$ (dotted line).

*Gent and Lindley* made a similar assumption in their analysis.
The filled circles are taken from Lindley/Teo's numerical solution for a Poisson's ratio of 0.49983 (the highest considered). The agreement with Equation 3 is good.

Experimental data (open circles) were obtained by compressing cylinders of unfilled NR vulcanisates of 25 mm diameter and various lengths using an Instron universal testing machine. The test pieces were first subjected to a few conditioning cycles between 0% and 12% strain. This was followed by incremental loading with load and displacement readings taken after 1 min of relaxation. The modulus was calculated from the slope of the least square line through this final loading curve between 0% and 5% strain. Replicate measurements usually gave results that agreed to within 5%. An estimate of Young's modulus was obtained by compressing cylinders between platsens lubricated with silicone grease. Other cylinders were bonded to rigid plates using cyanoacrylate adhesive. The experimental data are in satisfactory agreement with Equation 3.

Figure 3 compares Equation 4 with experimental data only since no other theoretical treatment is available. Cylinders were bonded to a rigid plate at one end using cyanoacrylate adhesive. Silicone grease was applied to the other end to lubricate that surface during compression. The experimental results are in reasonably good agreement with Equation 4.
INTERNAL HYDROSTATIC PRESSURE/TENSION

In deriving Equation 1, Gent and Lindley\textsuperscript{1} assumed that the free surface of the compressed cylinder bulges out to take the shape of a parabola. Their analysis predicts the existence of a hydrostatic pressure in the rubber. This pressure varies radially from zero at the perimeter to a maximum at the centre but does not vary axially. If the bonded cylinder is subjected to uniaxial tension instead of compression this internal pressure becomes a hydrostatic tension which accounts for internal cavitation\textsuperscript{6} that occurs when bonded rubber discs are subjected to tension.

Although Gent and Lindley’s treatment accounts for the phenomenon of cavitation in general, it fails to account for a small detail — for thick discs, internal cavitation invariably occurs near the bonded ends only\textsuperscript{8}. This indicates that the hydrostatic tension does in fact vary axially, being highest near the bonded ends.

The present treatment involving a three-part cylinder is somewhat better in accounting for this aspect of the cavitation phenomenon. The central portion of the cylinder being in simple, homogeneous tension/compression is free from internal tension/pressure. The internal hydrostatic tension/pressure is concentrated in the end-portions which bulge parabolically.

The difficulty with the present model lies in that parabolic end-portions would require the internal tension/pressure changing suddenly from zero in the central part of the cylinder to a finite and constant value in the end-portions. This abrupt change is unlikely and it appears that the assumption of parabolic end-portions must, strictly, be incorrect even though the success of Equations 3

![Figure 3. Comparison of Equation 4 with experiment.](image-url)
and 4 suggests that it is a good approximation.

CONCLUSIONS

Using a simple, approximate method relations have been derived that enable the evaluation of the compression modulus of tall cylinders which are either bonded at both ends or bonded at one end lubricated at the other. The relations obtained are in agreement with Lindley and Teo's numerical solution and with experimental data.

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