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Double Covariance Analysis in Manurial Experiments on Hevea

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The value of double covariance analysis using pre-treatment yield and girth as simultaneous calibrating variates for improving the precision of post-treatment yield comparison in fertiliser trials of Hevea *is assessed in relation to the use of individual calibrating variates in a single covariance analysis. Judicious use of covariance analysis is shown to improve precision of experiment in at least the initial three to four years of treatment.*

Choice of the appropriate calibrating variate to be used being dependent upon the circumstances of the experiment, it is desirable to obtain adequate records of both initial yield and girth before treatment application prior to the application of differential treatments.

The value of covariance technique for increasing the precision of treatment comparisons in fertiliser trials of *Hevea* has already been reviewed (NARAYANAN, 1966 and 1968) in relation to the usefulness of pre-treatment yield and girth as single calibrating variates. These have been used as simultaneous calibrating variates in this study.

EXPERIMENTAL

Ten manuring experiments previously subjected to covariance on girth were utilised for this study *(Table 1)*.

Yield recordings were made in respect of only 25–30 trees centrally located in each plot, but girth measurements were made on a larger number. In all experiments, the plots were provided with 'guards' to prevent inter-root competition. The plots varied from 0.5 to 1.0 acre.

The change in precision resulting from using pre-treatment yield (x_1) and girth (x_2) as simultaneous calibrating variates was expressed as the ratio $Vy/Vy.x_1x_2$, of the error variance of the unadjusted yield (Vy) to the effective error mean square of the adjusted yield $(Vy.x_1x_2)$. STEEL AND TORRIE (1960) have shown:

$$Vy.x_1x_2 = V'y.x_1x_2 \left[1+\frac{T_{11}E_{22}-2T_{12}E_{12}+T_{22}E_{11}}{(t-1)(E_{11}E_{22}-E_{12}^2)} \right]$$

where $V^1y.x_1x_2$ is the error variance of the adjusted yield.

T_{11} is the treatment sum of squares of x_1.

T_{22} „ „ „ „ „ of x_2.

T_{12} is the treatment sum of products of x_1 and x_2.

E_{11} is the residual sum of squares of x_1.

E_{22} „ „ „ „ „ of x_2.

E_{12} is the residual sum of products of x_1 and x_2.

$(t-1)$ is the degrees of freedom for treatments.

The partial regression coefficients b_1 (*i.e.*, $byx_1.x_2$), measuring the change in y per unit change of x_1 for any fixed value of x_2 and b_2 (*i.e.*, $byx_2.x_1$) measuring the change in y for unit change of x_2, keeping x_1 constant and their standard errors were also calculated (STEEL AND TORRIE, 1960). *Table A (Appendix)* shows the various steps involved in the procedure, and an experiment has been taken to illustrate the calculations needed for the analysis of double covariance.

RESULTS

Table 2 compares the changes in precision on post-treatment yield data obtained by the simultaneous use of initial yield and girth as calibrating variates $(Vy/Vy.x_1x_2)$ with the changes in precision obtained by using initial yield $(Vy/Vy.x_1)$ (NARAYANAN, 1966) and girth

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TABLE 1. EXPERIMENTAL DETAILS AND PRE-TREATMENT RECORDS*

Experiment	Plot size (acre)	Planting material	Design	First application of fertilisers	Pre-treatment records	
					Girth	Yield
A	0.82 7 rows of 21 points	PB 86	3^3 NPK factorial in single replicate —blocks of nine treatments	May '58	May '58	Dec '57— April '58
B	1.00 9 rows of 20 points	PB 86	3^3 NPK factorial in single replicate —blocks of nine treatments	May '59	Oct '58	Oct '58— April '59
C	0.54 7 rows of 14 points	PB 86	2^5 MgMnCuXY factorial in single replicate—blocks of eight treatments	Feb '60	Feb '60	Dec '59— Feb '60
D	Approx. 0.62 8 rows of 14 points	GI 1	2^5 NPKCu and type of P factorial in single replicate—blocks of eight treatments	Feb '60	Feb '60	Dec '59— Jan '60
E	0.49 7 rows of 12 points	PB 86	2^5 NPMgCu and type of P factorial in single replicate—blocks of eight treatments	Feb '60	Feb '60	Jan— Feb '60
F†	0.80 contour planting	RRIM 501	2^4 NPKMg factorial in three replications—blocks of eight treatments	May '59	Aug '59	Sept '58— April '59
G	0.21 1 row of 35 trees	RRIM 501	3^2 CuK factorial in two replications for each of the three clones—treatments are blocked in clones	Sept '58	March '58	July— Aug '58
J	0.73 6 rows of 22 points	PB 86	2^5 PKMgMn and type of P-factorial in single replicate—blocks of eight treatments	Feb '60	Jan '60	Jan— Feb '60
K	0.47 9 rows of 10 points	PB 86	3^3 NKMg factorial in single replicate —blocks of nine treatments	March '62	Jan '62	Jan— March '62
L	0.50 contour planting	Tjir 1	3^3 NMgMn factorial in single replicate— blocks of nine treatments	Aug '61	Aug '61	Aug '61

Note: †It is assumed that the girth at August 1959(taken here as pre-treatment)remains unaffected by the first treatment application in May 1959.

*For further details, see *Table 1*, NARAYANAN (1968).

TABLE 2. PRECISIONS USING SINGLE AND DOUBLE COVARIANCE ANALYSES OF PRE-TREATMENT YIELD AND GIRTH ON POST-TREATMENT YIELD

Experiment	Post-treatment data	Yield (tahils/tree/tapping)				Remarks
		1st year	2nd year	3rd year	4th year	
A	$Vy/Vy.x_1$	2.63	2.17	2.15	1.10	*Vy* based on 15 d.f.
	$Vy/Vy.x_2$	0.98	1.22	1.99	1.11	
	$Vy/Vy.x_1x_2$	2.14	1.77	2.56	0.98	
B	$Vy/Vy.x_1$	1.10	1.21	1.06	0.99	*Vy* based on 15 d.f.
	$Vy/Vy.x_2$	0.98	1.12	1.38	1.13	
	$Vy/Vy.x_1x_2$	1.01	1.23	1.33	1.05	
C	$Vy/Vy.x_1$	1.75	1.66	1.15	1.46†	*Vy* based on 22 d.f. or 13 d.f.
	$Vy/Vy.x_2$	1.43	1.13	1.45	1.50†	
	$Vy/Vy.x_1x_2$	2.39	1.71	1.39	1.82†	
D	$Vy/Vy.x_1$	1.62	3.42			*Vy* based on 14 d.f.
	$Vy/Vy.x_2$	4.37	2.82			
	$Vy/Vy.x_1x_2$	4.68	8.06			
E	$Vy/Vy.x_1$	1.55	0.99	0.96		*Vy* based on 14 d.f.
	$Vy/Vy.x_2$	1.12	0.99	1.03		
	$Vy/Vy.x_1x_2$	1.28	0.83	0.84		
F	$Vy/Vy.x_1$	2.27	2.07	1.48		*Vy* based on 32 d.f.
	$Vy/Vy.x_2$	1.13	1.47	1.22		
	$Vy/Vy.x_1x_2$	2.18	2.26	1.49		
G‡	$Vy/Vy.x_1$	2.57	1.21	1.28	0.96	*Vy* based on 24 d.f.
	$Vy/Vy.x_2$	0.93	0.97	1.16	1.02	
	$Vy/Vy.x_1x_2$	2.39	1.14	1.41	0.97	
J	$Vy/Vy.x_1$	0.86	0.86	0.88	0.92	*Vy* based on 14 d.f.
	$Vy/Vy.x_2$	1.41	1.37	1.27	1.16	
	$Vy/Vy.x_1x_2$	1.21	1.20	1.08	1.02	
K	$Vy/Vy.x_1$	1.13	0.90	0.88	0.85	*Vy* based on 15 d.f.
	$Vy/Vy.x_2$	1.12	2.15	1.88	1.96	
	$Vy/Vy.x_1x_2$	1.24	1.88	1.61	1.70	
L	$Vy/Vy.x_1$	0.92	0.96	0.90	0.96	*Vy* based on 15 d.f.
	$Vy/Vy.x_2$	0.96	1.62	1.11	1.15	
	$Vy/Vy.x_1x_2$	0.88	1.58	0.99	1.10	

† Does not cover one full year. ‡ Yield has been expressed in g/tree/tapping.

Vy denotes the error variance of the unadjusted yield.
$Vy.x_1$ denotes the effective error mean square of the adjusted yield using yield as a single covariate.
$Vy.x_2$,, ,, ,, ,, ,, ,, ,, ,, ,, ,, ,, girth ,, ,, ,, ,,
$Vy.x_1x_2$,, ,, ,, ,, ,, ,, ,, ,, ,, ,, ,, both yield and girth as double covariates.

($Vy/Vy.x_2$) (NARAYANAN, 1968) as single co-variates. The regression coefficients and their standard errors are similarly brought together in *Tables 3 (a)* and *(b)*.

In Experiment A, the double covariance using yield and girth increased its precision 2.14 times in the first year, 1.77 times in the second year and 2.56 times in the third year. The partial regression coefficient b_1 is significant for the first three years, while the partial regression coefficient b_2 is significant only in the third year.

In Experiment B, only moderate increases in precision were obtained in the first four years (in the range of 1.01–1.33). The regression coefficient b_1 was significant only in the second year while the regression coefficient b_2 was significant in the third and fourth years.

The increased precision resulting from the use of double covariance varied between 1.39 and 2.39 in first four years for Experiment C. The regression coefficient b_1 is significant in all the years except the third, whereas the regression coefficient b_2 is significant in all the years except the second.

In Experiment D, which continued only for two years, the increased precisions were of the order of 4.68 in the first year and 8.06 in the second year. The regression coefficient b_1 showed significance only in the second year, while b_2 showed significance in both the years.

Increased precision due to double covariance was shown only in the first year for Experiment E. Only the regression coefficient b_1 was shown to be significant in the first year, whereas b_2 was not significant in the first year.

In experiment F, which lasted for three years, the increased precisions in yield due to double covariance varied from 2.18 in the first year, 2.26 in the second and 1.49 in the third years. The partial regression coefficient b_1 was significant in all the three years, while the coefficient b_2 was significant only in the second year.

The precisions using both yield and girth as double covariates increased in the first three years for Experiment G. The partial regression coefficient b_1 was significant in the first three years, while the regression coefficient b_2 showed significance only in the third year.

For Experiments J and K, and for the second and fourth years of L, increased precisions in yield due to double covariance were shown in the first four years. The partial regression coefficient b_1 is not significant in any of the years in any of the experiments (except in the first year for Experiment K), while the partial regression coefficient b_2 was significant in almost all instances.

DISCUSSION

It was thought worthwhile to compare at the outset these results with those obtained in single covariance analysis using either of the two calibrating variates—yield or girth. In the thirty-six cases (experiments by years), precision was reliably increased in twenty-one instances by simple covariance on pre-treatment yield and in twenty-eight instances using pre-treatment girth: in seventeen cases both pre-treatment yield and girth independently were effective calibrating variates [see *Table 3 (a)*]. In twenty-eight instances the double covariance was effective, it being distinctly more effective than either of the calibrating variates taken singly (see *Table 2*) in nine instances or cases. The single covariance on pre-treatment yield was the most effective in seven instances and on pre-treatment girth in thirteen. Pre-treatment yield was generally the more effective single calibrating variate, though it was quite ineffective in Experiments J, K and L; pre-treatment girth was of little value in Experiment G. Improvement was confined to the first year in Experiment E, but benefits often were obtained over the initial three to four years in the other experiments.

It can thus be seen that increased precisions in the post-treatment yield records can be obtained by the utilisation of either the initial yield or girth or both. Whether the post-treatment yields are better adjusted by pre-treatment yield or girth appears to depend upon circumstances (PEARCE AND BROWN, 1960). In Experiments A, E, F and G, initial yield usually was a better calibrating variate than trunk girth, while in Experiments J, K and L, trunk girth alone was effective. In Experiments B, C and D, both yield and girth seem to offer as two calibrating variates. In four of these ex-

TABLE 3A. SINGLE COVARIANCE ANALYSIS: REGRESSION COEFFICIENTS AND THEIR STANDARD ERRORS

Experiment	Post-treatment data	Yield (tahils/tree/tapping)							
		1st year		2nd year		3rd year		4th year	
A	$b_1 \pm S.E._1$	0.687±0.134	***	0.680±0.134	***	0.785±0.179	***	0.462±0.268	→(P <0.10)
	$b_2 \pm S.E._2$	0.0010±0.0006	N.S.	0.0014±0.0006	*	0.0022±0.0005	***	0.0016±0.0007	*
B	$b_1 \pm S.E._1$	0.418±0.222	(P <0.10)	0.443±0.193	*	0.464±0.276	→(P <0.10)	0.292±0.223	N.S.
	$b_2 \pm S.E._2$	0.0014±0.0010	N.S.	0.0018±0.0008	(P <0.10)	0.0030±0.0010	→**	0.0019±0.0009	*
C	$b_1 \pm S.E._1$	0.693±0.156	***	0.685±0.163	***	0.591±0.281	→*	0.924±0.312	* †
	$b_2 \pm S.E._2$	0.0015±0.0005	**	0.0011±0.0005	→*	0.0025±0.0008	**	0.0032±0.0010	** †
D	$b_1 \pm S.E._1$	0.764±0.220	**	1.051±0.166	***				
	$b_2 \pm S.E._2$	0.0040±0.0005	***	0.0041±0.0007	***				
E	$b_1 \pm S.E._1$	0.672±0.202	**	0.411±0.265	N.S.	0.235±0.169	N.S.		
	$b_2 \pm S.E._2$	0.0033±0.0016	(P <0.10)	0.0026±0.0018	N.S.	0.0018±0.0011	N.S.		
F	$b_1 \pm S.E._1$	0.668±0.102	***	0.724±0.119	***	1.038±0.249	***		
	$b_2 \pm S.E._2$	0.0029±0.0012	*	0.0047±0.0012	***	0.0066±0.0012	**		
G‡	$b_1 \pm S.E._1$	0.722±0.114	***	0.458±0.175	→**	0.767±0.259	**	0.213±0.238	N.S.
	$b_2 \pm S.E._2$	0.0175±0.0298	N.S.	0.0349±0.0309	N.S.	0.1050±0.0433	*	0.0586±0.0365	N.S.
J	$b_1 \pm S.E._1$	0.002±0.246	N.S.	0.031±0.224	N.S.	−0.163±0.261	N.S.	−0.461±0.453	N.S.
	$b_2 \pm S.E._2$	0.0024±0.0009	*	0.0022±0.0008	*	0.0024±0.0010	*	0.0037±0.0019	(P <0.10)
K	$b_1 \pm S.E._1$	0.625±0.293	(P <0.10)	0.392±0.444	N.S.	0.343±0.511	N.S.	0.022±0.518	N.S.
	$b_2 \pm S.E._2$	0.0024±0.0012	(P <0.10)	0.0053±0.0012	***	0.0057±0.0014	**	0.0058±0.0014	→***
L	$b_1 \pm S.E._1$	0.075±0.106	N.S.	0.103±0.100	N.S.	0.062±0.170	N.S.	0.127±0.125	N.S.
	$b_2 \pm S.E._2$	0.0014±0.0011	N.S.	0.0029±0.0008	**	0.0033±0.0017	(P <0.10)	0.0027±0.0013	→*

† Does not cover one full year. ‡ Yield has been expressed in g/tree/tapping. ***P <0.001 **P <0.01 *P <0.05 N.S. : Not significant

b_1 is the regression coefficient using yield as the single covariate. b_2 is the regression coefficient using girth as the single covariate.

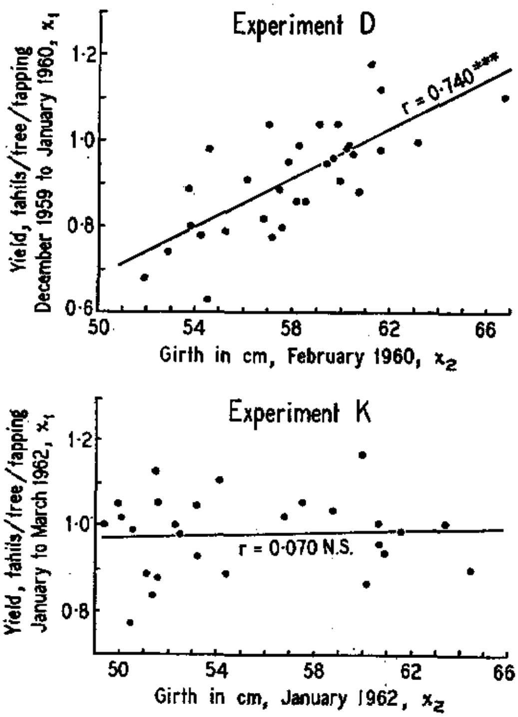
Arrow (→) indicates 'tending to'.

TABLE 3B. DOUBLE COVARIANCE ANALYSIS: PARTIAL REGRESSION COEFFICIENTS AND THEIR STANDARD ERRORS

Experiment	Post-treatment data	Yield (tahils/tree/tapping)							
		1st year		2nd year		3rd year		4th year	
A	$b_1 \pm$S.E.$_1$	0.766±0.172	***	0.589±0.198	*	0.480±0.191	*	0.191±0.329	N.S.
	$b_2 \pm$S.E.$_2$	−0.0004±0.0005	N.S.	0.0004+0.0006	N.S.	0.0014±0.0005	*	0.0012±0.0009	N.S.
B	$b_1 \pm$S.E.$_1$	0.349±0.235	N.S.	0.349±0.193	(P <0.10)	0.270±0.250	N.S.	0.170±0.220	N.S.
	$b_2 \pm$S.E.$_2$	0.0009±0.0010	N.S.	0.0013±0.0008	N.S.	0.0026±0.0011	*	0.0017±0.0009	(P <0.10)
C	$b_1 \pm$S.E.$_1$	0.578±0.137	***	0.617±0.164	→***	0.352±0.257	N.S.	0.649±0.280	* †
	$b_2 \pm$S.E.$_2$	0.0011±0.0004	**	0.0006±0.0004	N.S.	0.0020±0.0008	*	0.0023±0.0009	* †
D	$b_1 \pm$S.E.$_1$	0.259±0.158	N.S.	0.688±0.132	***				
	$b_2 \pm$S.E.$_2$	0.0034±0.0006	***	0.0024±0.0005	***				
E	$b_1 \pm$S.E.$_1$	0.638±0.285	*	0.290±0.370	N.S.	0.100±0.231	N.S.		
	$b_2 \pm$S.E.$_2$	0.0004±0.0019	N.S.	0.0012±0.0025	N.S.	0.0014±0.0016	N.S.		
F	$b_1 \pm$S.E.$_1$	0.676±0.123	***	0.579±0.135	***	0.848±0.294	**		
	$b_2 \pm$S.E.$_2$	−0.0001±0.0010	N.S.	0.0022±0.0011	→*	0.0029±0.0024	N.S.		
G‡	$b_1 \pm$S.E.$_1$	0.735±0.119	***	0.431±0.182	*	0.651±0.249	*	0.137±0.239	N.S.
	$b_2 \pm$S.E.$_2$	−0.0912±0.1891	N.S.	0.1931±0.2900	N.S.	0.8418±0.3965	*	0.5359±0.3809	N.S.
J	$b_1 \pm$S.E.$_1$	0.107±0.205	N.S.	0.126±0.188	N.S.	−0.066±0.234	N.S.	−0.318±0.428	N.S.
	$b_2 \pm$S.E.$_2$	0.0025±0.0009	*	0.0023±0.0009	*	0.0023±0.0011	*	0.0034±0.0019	→(P <0.10)
K	$b_1 \pm$S.E.$_1$	0.548±0.275	(P <0.10)	0.193±0.302	N.S.	0.125±0.372	N.S.	−0.206±0.361	N.S.
	$b_2 \pm$S.E.$_2$	0.0020±0.0011	(P <0.10)	0.0052±0.0012	***	0.0056±0.0015	**	0.0059±0.0015	**
L	$b_1 \pm$S.E.$_1$	0.066±0.105	N.S.	0.085±0.075	N.S.	0.041±0.157	N.S.	0.111±0.113	N.S.
	$b_2 \pm$S.E.$_2$	0.0013±0.0012	N.S.	0.0029±0.0008	**	0.0033±0.0018	(P <0.10)	0.0026±0.0013	→*

†Does not cover one full year. ‡Yield has been expressed in g/tree/tapping. ***: P <0.001 **: P <0.01 * : P <0.05 N.S. : Not significant
b_1 is the partial regression coefficient using yield in double covariance. b_2 is the partial regression coefficient using girth in double covariance.
Arrow (→) indicates 'tending to'.

periments (B, D, F and J), linear correlations between initial yield and girth were significant (see *Table 3*, NARAYANAN, 1968). These correlations depend to a certain extent on the length of the initial yield records used, but it can be said that the pre-treatment yield and girth act as uncorrelated variates in some cases but not in others [*Figures 1(a)* and *1(b)*].



Thus, in any particular instance, a double covariance using both the initial yield and girth is desirable, for either or both of the variates may be effective. For a double covariance to be better than a single covariance of either variates, it appears necessary that both covariates should be effective in reducing the experimental error. A greater increase in precision was obtained by applying double covariance than by applying the *more effective* of the calibrating variates individually only where the *less effective* of the calibrating variates applied individually increased precision by more than 22%. Where this increase in precision was smaller, a double covariance seldom held much advantage, particularly as the degrees of freedom for the error term is reduced by two in double covariance compared to one in the case of single covariance. Where one of the covariance is ineffective, the double covariance may not be expected to give good results, but the effectiveness of each can only be determined by double covariance.

Pre-treatment yield is most effective as a calibrating variate in the early years of an experiment but its effectiveness declines progressively with time; the effectiveness of girth is much better sustained, often increasing over the first three years. Double covariance sustains a more consistent improvement over the first three years than either calibrating variate used singly, but with the declining effectiveness of yield it becomes less effective in the fourth year. Omitting Experiment D, in the second year of which double covariance resulted in an inexplicably large improvement in experimental precision, the increase in precision by applying the most effective covariance (either single or double, whichever is higher) averages about 1.7, giving about 23% reduction in the standard error of a treatment mean.

The effectiveness of pre-treatment yield as a calibrating variate depends partly on the length of time over which the pre-treatment records were taken. For Experiments J, K and L, this length of record is only one to three months which seems very inadequate. In the other experiments, initial yield records were between two to eight months. Greater precision might have been achieved in all experiments if the initial yield records had lasted a year or so longer.

CONCLUSIONS

Though double covariance of pre-treatment yield and girth will not necessarily further increase the precision of post-treatment comparisons obtained by using one or other of these calibrating variates separately, the choice of the analysis to be used so depends on the circumstances of the experiment that wherever possible it is desirable to collect adequate records of both before application of differential treatments. A material improvement in precision, the value of which has already been pointed out (NARAYANAN, 1966), will then be secured at least for the first three or four years of treatment by applying the more effective of the possible covariance analyses.

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APPENDIX

Herein are set out in a tabular form the details of the various steps involved in the procedure of the analysis of double covariance. Experiment A has been chosen as an example to illustrate the various calculations needed for the purpose.

Experiment A is a 3^3 NPK factorial trial in single replicate of three blocks of nine treatments each. The first application of fertiliser was made in May 1958, and the pre-treatment yield (x_1) and girth (x_2) records were obtained during December 1957 to April 1958 and in May 1958 respectively. The post-treatment yield (y) has been taken for the third year (May 1960–April 1961).

TABLE A. EXPERIMENT A: ANALYSES OF VARIANCE AND COVARIANCE

Source	d.f.	Sums of squares and products					
		x_1^2	x_2^2	x_1x_2	y^2	yx_1	yx_2
Blocks	2	0.026403	6 047.29	−7.2526	0.014200	0.001494	7.0603
N	2	0.000908	941.82	0.0482	0.006804	−0.002483	−0.0332
P	2	0.000655	5 018.32	1.7303	0.017203	0.002528	8.5729
K	2	0.010854	201.64	0.2043	0.004523	−0.004219	0.6946
N′P′*	1	0.000096	717.65	−0.2578	0.000363	0.000187	−0.5005
N′K′	1	0.005167	3 884.40	4.4820	0.023763	0.011081	9.6120
P′K′	1	0.000884	1 456.40	−1.1416	0.001302	−0.001073	1.3854
Error	15	0.077422	9 873.20	16.9336	0.082598	0.060769	21.9018
Total	26	0.122389	28 140.72	14.7464	0.150756	0.068284	48.6933

*N′P′, N′K′ etc refer to linear interactions.

Using error line, the regression coefficients for single covariance are obtained as

$$b_1 = \frac{\Sigma yx_1}{\Sigma x_1^2} = \frac{0.060769}{0.077422} = 0.785$$

$$b_2 = \frac{\Sigma yx_2}{\Sigma x_2^2} = \frac{21.9018}{9873.20} = 0.002218$$

The partial regression coefficients b_1 and b_2 for double covariance are obtained by solving the two simultaneous equations given below:

$$\left.\begin{aligned}(\Sigma x_1^2)b_1+(\Sigma x_1x_2)b_2 &= \Sigma x_1y\\ (\Sigma x_1x_2)b_1+(\Sigma x_2^2)b_2 &= \Sigma x_2y\end{aligned}\right\} \text{—(A)}$$

b_1 and b_2 are abbreviated for $byx_1.x_2$ and $byx_2.x_1$.

Using error line, the equations are

$$0.077422\, b_1+16.9336\, b_2 = 0.060769$$
$$16.9336\, b_1+9873.20\, b_2 = 21.9018$$

Solving for b_1 and b_2 from above,

$$b_1 = 0.4796,$$
$$b_2 = 0.001396.$$

$$Vy = \frac{\Sigma y^2}{15} = 0.005507$$

$$V'y.x_1 = \frac{\Sigma y^2 - b_1\, \Sigma x_1y}{14} = 0.002492$$

$$V'y.x_2 = \frac{\Sigma y^2 - b_2\, \Sigma x_2y}{14} = 0.002430$$

$$V'y.x_1x_2 = \frac{\Sigma y^2 - byx_1.x_2\;\Sigma x_1y - byx_2.x_1\;\Sigma x_2y}{13} = 0.001760.$$

For single covariance,

$$Vy.x_1 = V'y.x_1 \left[1+\frac{T_{11}}{(t-1)\,E_{11}}\right] = 0.002558.$$

Similarly,

$$Vy.x_2 = V'y.x_2 \left[1+\frac{T_{22}}{(t-1)\,E_{22}}\right]$$

For double covariance,

$$Vy.x_1x_2 = V'y.x_1x_2 \left[1+\frac{T_{11}E_{22}-2T_{12}E_{12}+T_{22}E_{11}}{(t-1)(E_{11}E_{22}-E_{12}^2)}\right] = 0.002152$$

(see text for explanation of these symbols).

Thus

$$Vy/Vy.x_1 = 2.15$$
$$Vy/Vy.x_2 = 1.99$$
$$Vy/Vy.x_1x_2 = 2.56$$

$$\text{S.E. of } b_1 = \sqrt{\frac{V\,y.x_1}{E_{11}}} = 0.1794$$

$$\text{S.E. of } b_2 = \sqrt{\frac{V\,y.x_2}{E_{22}}} = 0.000496$$

Rewriting the equation (A) in the form

$$\left.\begin{aligned}(\Sigma x_1^2)C_{11}+(\Sigma x_1x_2)C_{12} &= 1\\ (\Sigma x_1x_2)C_{11}+(\Sigma x_2^2)C_{12} &= 0\end{aligned}\right\}$$

and

$$\left.\begin{aligned}(\Sigma x_1^2)C_{21}+(\Sigma x_1x_2)C_{22} &= 0\\ (\Sigma x_1x_2)C_{21}+(\Sigma x_2^2)C_{22} &= 1\end{aligned}\right\}$$

Solving for the C's,

$$C_{11} = 20.6701$$
$$C_{12} = -0.035451$$
$$C_{21} = -0.035451$$
$$C_{22} = 0.000162$$

The partial regression coefficients $byx_1.x_2$ and $byx_2.x_1$ can also be obtained as follows:

$$byx_1.x_2 = C_{11}\,\Sigma x_1y + C_{12}\,\Sigma x_2y$$
$$byx_2.x_1 = C_{21}\,\Sigma x_1y + C_{22}\,\Sigma x_2y$$

$$\text{S.E. of } byx_1.x_2 = \sqrt{V'y.x_1x_2} \times \sqrt{C_{11}} = 0.1907$$

$$\text{S.E. of } byx_2.x_1 = \sqrt{V'y.x_1x_2} \times \sqrt{C_{22}} = 0.000480.$$